INTERACTION OF TWO WATER-BEARING HORIZONS SEPARATED BY A SLIGHTLY PERMEABLE BAND

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 2, pp. 152-155, 1967

Consider the unsteady axially symmetric filtration in a stratified aquifer consisting of two highly permeable strata separated by a band of low permeability. The upper stratum has a cover of low permeability (cover rock) containing the water table, while an impermeable layer lies under the lower one (see figure). The filtration is everywhere elastic. An analogous problem has been considered [1-5] either subject to the condition that the pressure in one stratum remains unaltered, or with the permeability of the cover rock neglected, or on the assumption that the passage through the band occurs under a high differential. Shestakov [6] considered this problem on the assumption that the cover rock is impermeable and obtained an approximate solution for the case in which the two strata are equal in permeability.

NOTATION

- r—radial coordinate,
- z—vertical coordinate,
- h—head reckoned from the cover,
- h_0—initial head,
- k—filtration coefficient,
- m—stratum thickness,
- $\langle m_1 \rangle$—mean thickness of cover,
- T—filtration permeability,
- μ°—elastic water-release factor,
- μ_1—release factor of cover,
- a—pressure-permeability factor,
- α_1—coefficient for transfer via cover,
- α_3—coefficient for transfer via band.

$$T = km, \quad a = T / \mu^\circ, \quad \alpha_1 = k_1 / \langle m_1 \rangle, \quad \alpha_3 = k_3 / m_3 .$$

The meaning of the numerical subscripts will be clear from the figure.

§1. The following are the linear differential equations derived on the assumption that the vertical component of the filtration rate is independent of z within the cover rock:

$$\mu_1 \frac{\partial h_1}{\partial t} = \alpha_1 (h_2 - h_1),$$

$$\mu_2^\circ \frac{\partial h_2}{\partial t} = T_2 \Delta h_2 - k_3 \left[\frac{\partial h_3}{\partial z}\right]_{z=-m_2} - \alpha_1 (h_1 - h_2),$$

$$\mu_3 \frac{\partial h_3}{\partial t} = T_3 \frac{\partial^2 h_3}{\partial z^2},$$

$$\mu_4^\circ \frac{\partial h_4}{\partial t} = T_4 \Delta h_4 + k_3 \left[\frac{\partial h_3}{\partial z}\right]_{z=-m_2-m_3}, \quad \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right). \tag{1.1}$$

Khantush [5] gave the third equation in this system, which describes the motion in the band under elastic conditions.

We assume that there is no motion at the start and that the perfect borehole collecting water from the lower stratum is of zero radius. Then the initial and boundary conditions are

$$h_1(r, 0) = h_2(r, 0) = h_3(r, z, 0) = h_4(r, 0) = h_0,$$

$$h_3(r, -m_2, t) = h_2(r, t),$$

$$h_3(r, -m_2-m_3, t) = h_4(r, t), \tag{1.2}$$

$$\lim_{r\to 0} 2\pi T_4 r \frac{\partial h_4}{\partial r} = -Q, \qquad \lim_{r\to 0} 2\pi T_2 r \frac{\partial h_2}{\partial r} = 0. \tag{1.3}$$

Here Q is the constant flow rate from the borehole. We apply a Laplace transformation to (1.1) to get

$$S_1 = S_2 \frac{\alpha_1}{p\mu_1 + \alpha_1}, \quad (S \doteqdot h_0 - h),$$

$$S_3 = \frac{S_2 \operatorname{sh}[\sigma(z + m_2 + m_3)] - S_4 \operatorname{sh}[\sigma(z + m_2)]}{\operatorname{sh}(\sigma m_3)}, \tag{1.4}$$

$$\left(\sigma = \left(\frac{p}{a_3}\right)^{1/2}\right), \tag{1.5}$$

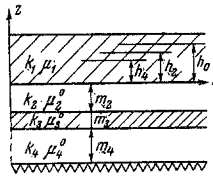
$$T_2 \Delta S_2 - \omega_2^2 S_2 + S_4 \alpha_3 \frac{\sigma m_3}{\operatorname{sh}(\sigma m_3)} = 0,$$

$$\omega_2^2 = p\mu_2^\circ + \alpha_1 \frac{p\mu_1}{p\mu_1 + \alpha_1} + \alpha_3 \sigma m_3 \operatorname{cth}(\sigma m_3),$$

$$T_4 \Delta S_4 - \omega_4^2 S_4 + S_2 \alpha_3 \frac{\sigma m_3}{\operatorname{sh}(\sigma m_3)} = 0,$$

$$\omega_4^2 = p\mu_4^\circ + \alpha_3 \sigma m_3 \operatorname{cth}(\sigma m_3). \tag{1.6}$$

Here S is the Laplace transform of the region of reduced pressure. The exact solution to (1.6) is very complicated, so we shall consider approximate solutions more suitable for calculation.



§2. Let t be the time for which water is taken from the stratum, which is not less than the smaller of 10 μ_1/α_1 and 10 μ_3°/α_3. In that case, parameter p should have values not less than 0.1 α_1/μ_1° and 0.1 × α_3/μ_3° in the region of the images, and we may put approximately that

$$\sigma m_3 \operatorname{cth}(\sigma m_3) = 1 + \frac{p\mu_3^\circ}{3\alpha_3}\quad \frac{\sigma\mu_3}{\operatorname{sh}(\sigma m_3)} = 1,$$

$$\frac{p\mu_1}{p\mu_1 + \alpha_1} = p\,\frac{\mu_1}{\alpha_1}, \quad \frac{\alpha_1}{p\mu_1 + \alpha_1} = 1. \tag{2.1}$$

Equations (1.4) and (1.6) then become

$$S_1 = S_2, \tag{2.2}$$

$$T_2 \Delta S_2 - w_2^2 S_2 + \alpha_3 S_4 = 0, \qquad \omega_2^2 = p\mu_2^* + \alpha_3,$$

$$T_4 \Delta S_4 - \omega_4^2 S_4 + \alpha_3 S_2 = 0, \qquad \omega_4^2 = p\mu_4^* + \alpha_3. \tag{2.3}$$

$$\mu_2^* = \mu_1 + \mu_2^\circ + {}^1/_3\,\mu_3^\circ, \qquad \mu_4^* = \mu_4^\circ + {}^1/_3\,\mu_3^\circ. \tag{2.4}$$

A solution to (2.3) that is bounded as $r \to \infty$ is sought in the form

$$S_2 = ACK_0(r\beta_1) + DBK_0(r\beta_2),$$

$$S_4 = CK_0(r\beta_1) + DK_0(r\beta_2). \tag{2.5}$$

Here $K_0(r\beta)$ is a modified zero-order Bessel function of the second kind,

$$A = 1 + \frac{p\mu_4^* - T_4\beta_1^2}{\alpha_3},$$

$$\beta^2 = \frac{1}{2T_2}\left\{p\left(\mu_2^* + \mu_4^* \frac{T_2}{T_4}\right) + \alpha_3\left(1 + \frac{T_2}{T_4}\right) + \sqrt{M}\right\},$$

$$B = \frac{1}{1 + (p\mu_2^* - T_2\beta_2^2)/\alpha_3},$$

$$M = 4\alpha_3^2 \frac{T_2}{T_4} + \left\{p\left(\mu_2^* - \mu_4^* \frac{T_2}{T_4}\right) + \alpha_3\left(1 - \frac{T_2}{T_4}\right)\right\}^2. \tag{2.6}$$

Constants C and D are defined by conditions (1.3), so the solution for large t is

$$S_2(r, p) = \frac{Q}{2\pi T_4 p}\frac{AB}{A-B}\{K_0(r\beta_2) - K_0(r\beta_1)\},$$

$$S_4(r, p) = \frac{Q}{2\pi T_4 p}\frac{1}{A-B}\{AK_0(r\beta_2 - BK_0(r\beta_1)\},$$

$$S_1(r, p) = S_2, (rp). \quad (2.7)$$

The following are particular examples in which the originals of the functions are given in terms of tabulated functions.

(1) Rocks are very inhomogeneous, so hydrogeologic parameters are of low accuracy. We assume that T_2 and T_4 are of the same order, as are μ_2° and μ_4°, so $T_2 = T_4 = T$, $\mu_2^\circ = \mu_4^\circ = \mu^\circ$, which with (2.7) gives

$$\beta^2 = (1/T)\{p(\mu^\circ + {}^1/_2\mu_1 + {}^1/_3\mu_3^\circ) + \alpha_3 \pm \alpha_3\sqrt{1 + p^2\mu_1^2/4\alpha_3^2}\}.$$

Since p is small, we have approximately that

$$\beta_1^2 = \frac{p}{a} + 2\frac{\alpha_3}{T}, \quad \beta_2^2 = \frac{p}{a}, \quad a = \frac{T}{\mu^\circ + {}^1/_2\mu_1 + {}^1/_3\mu_3^\circ},$$

$$\frac{A}{A-B} = \frac{1}{2}, \quad \frac{AB}{A-B} = \frac{1}{2}, \quad \frac{B}{A-B} = -\frac{1}{2}, \quad (2.8)$$

and (2.7) becomes

$$S_2 = \frac{Q}{4\pi Tp}\{K_0(r\beta_2) - K_0(r\beta_1)\},$$

$$S_4 = \frac{Q}{4\pi Tp}\{K_0(r\beta_2) + K_0(r\beta_1)\}. \quad (2.9)$$

Operational calculus gives us the originals:

$$p^{-1}K_0(r\beta_1) \doteqdot {}^1/_2 W(u, \delta), \quad p^{-1}K_0(r\beta_2) \doteqdot -{}^1/_2 \,\mathrm{Ei}(-u)$$

$$-\mathrm{Ei}(-u) = \int_u^\infty e^{-\lambda}\frac{d\lambda}{\lambda}, \quad W(u, \delta) = \int_u^\infty \exp\left(-\lambda - \frac{\delta^2}{4\lambda}\right)\frac{d\lambda}{\lambda}$$

$$\left(u = \frac{r^2}{4at}, \quad \delta = r\left(\frac{2\alpha_3}{T}\right)^{1/2}\right).$$

Functions $W(u, \delta)$ has been tabulated [5]. These relationships give

$$h_1 = h_2, \quad h_2 = h_0 - \frac{Q}{8\pi T}\{-\mathrm{Ei}(-u) - W(u, \delta)\},$$

$$h_4 = h_0 - \frac{Q}{8\pi T}\{-\mathrm{Ei}(-u) + W(u, \delta)\}. \quad (2.10)$$

In particular, if the cover rock is taken as impermeable, the relation for h_1 drops out. Putting $\mu_1 = 0$ in (2.8), we get the solution given by Shestakov [6], which he found by replacing the equations for the band by a finite-difference equation, though he put $\alpha = 1/2$ instead of $\alpha = 1/3$ in the expression $a = T/(\mu^\circ + \alpha\mu_3^\circ)$.

The pressure difference between the two strata is

$$h_2(r, t) - h_4(r, t) = \frac{Q}{4\pi T}W(u, \delta),$$

and for this we have the limiting relation

$$[h_2(r, t) - h_4(r, t)]_{t=\infty} = \frac{Q}{2\pi T}K_0(\delta). \quad (2.11)$$

The non-steady state part of the difference becomes unimportant for $u < 0.10\,\delta^2$. From the limiting relation

$$\lim_{p\to 0} pF(p) = f(\infty) \qquad (F(p) \doteqdot f(t)),$$

we get from (2.8) that (2.11) retains its form for other permeabilities of the strata if we assume that in the general case

$$\delta = r\left(\alpha_3\frac{T_2 + T_4}{T_2 T_4}\right)^{1/2}. \quad (2.12)$$

(2) We assume that the water is taken from the upper stratum, with $T_4 \gg T_2$ ($T_4 \to \infty$), which gives us a scheme corresponding to the case in which the lower stratum (containing more water) has a head h_4 = const, without reference to the borehole. Putting $T_4 \to \infty$ in (2.6), we have

$$\beta_1^2 = \frac{p}{a} + \frac{\alpha_3}{T_2}, \quad \beta_2^2 = 0, \quad \frac{A}{A-B} = 1, \quad (2.13)$$

and so the solution becomes

$$h_1 = h_2, \quad h_2 = h_0 - \frac{Q}{4\pi T_2}W(u, \delta)$$

$$\left(u = \frac{r^2}{4at}, \quad a = \frac{T_2}{\mu_2^*}, \quad \delta = r\left(\frac{\alpha_3}{T_2}\right)^{1/2}\right). \quad (2.14)$$

In the steady state

$$h_2(r) = h_0 - \frac{Q}{2\pi T_2}K_0(\delta). \quad (2.15)$$

§4. If t does not exceed the smaller of $0.1\,\mu_3^\circ/\alpha_3$ and $0.1\,\mu_1/\alpha_1$ and corresponds to a value of p not less than the larger of $10\,\alpha_3/\mu_3^\circ$ and $10\,\alpha_1/\mu_1$, then approximately

$$\frac{\sigma m_3}{\mathrm{sh}(\sigma m_3)} = 0, \quad \frac{p\mu_1}{p\mu_1 + \alpha_1} = 1,$$

$$\frac{\alpha_1}{p\mu_1 + \alpha_1} = 0, \quad \alpha_1 = 0. \quad (3.1)$$

Then (1.4) and (1.6) become

$$S_1 = 0, \quad \Delta S_2 - \omega_2^2 S_2 = 0, \quad \omega_2^2 = T_2^{-1}(p\mu_2^\circ + k_3\sqrt{p/a_3}),$$

$$\Delta S_4 - \omega_4^2 S_4 = 0, \quad \omega_4^2 = T_4^{-1}(p\mu_4^\circ + k_3\sqrt{p/a_3}). \quad (3.2)$$

The solution, bounded as $r \to \infty$, that satisfies (1.2) and (1.3) is

$$S_1 = S_2 = 0, \qquad S_4 = \frac{Q}{2\pi T_2 p}K_0(r\omega_4). \quad (3.3)$$

We note that

$$\frac{1}{p}K_0(r\omega_4) =$$

$$= \frac{1}{2}\int_u^\infty e^{-\lambda}\,\mathrm{erfc}\left(\frac{\nu\sqrt{u}}{\sqrt{\lambda(\lambda-u)}}\right)\frac{d\lambda}{\lambda} = \frac{1}{2}H(u, \nu)$$

$$\left(u = \frac{r^2}{4a_4 t}, \quad a_4 = \frac{T_4}{\mu_4^\circ}, \quad \nu = \frac{rk_3}{4T_4}\left(\frac{a_4}{a_3}\right)^{1/2}\right). \quad (3.4)$$

Function $H(u, \nu)$ has been tabulated [5], so the solution for small t is

$$h_1 = h_2 = h_0, \quad h_4(r, t) = h_0 - \frac{Q}{4\pi T_4}H(u, \nu). \quad (3.5)$$

This solution indicates the importance of considering the elastic mode of filtration in the main aquifers and in the band of low permeability. The leakage through that band substantially reduces the pressure drop in the stratum used for extraction, since the transfer of pressure from one stratum to the other occurs with a time delay. This means that t must not be less than that given by (3.1) if pumping from a borehole is performed in order to detect interaction of strata and to determine hydrogeologic parameters.

For instance, if $k_1 = 0.1$ m/day, $k_3 = 10^{-4}$ m/day, $\langle m_1\rangle = 100$ m, $m_3 = 10$ m, $\mu_1 = 0.1$, $\mu_3^\circ = 10^{-4}$, then (3.1) indicates that a reduction in the pressures is to be expected not sooner than a day after the start of pumping.

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12 September 1966 Tashkent